

## Exploring The Logistic Population Model As A Discrete Dynamical System In Calculus Using The TI - 92

Mary Ann Connors

Department of Mathematical Sciences

United States Military Academy

West Point, NY 10996

maryann-connors@usma.edu

We begin with explorations examining the long term behavior of some population models to discover the need for terminology and theory for studying limits. Students have to do plenty of computing to see populations that grow without bound, populations that always approach a limit, and some that approach cyclic behavior.

Suppose the population of a species of temperate zone insects numbers 2,000 in 1996 and 2,200 in 1997. The ratio of the current generation to the previous generation, 1.1, indicates that the net population growth for the first year is 10%. Assuming the growth rate will be the same every year, what will the population figures be for several years?

This type of model is described by an equation of the form  $p_{n+1} = R p_n$ , where  $R$  is a constant called the **population multiplier**. After experimenting, students notice that when  $R > 1$ , the population increases, when  $R = 1$ , the population remains constant, and when  $R < 1$ , the population decreases.

Let's consider another model. Suppose a species lives in an unfavorable habitat. Although 20% die in a given generation, 100 more of the species immigrate to the habitat at the end of each generation. Will this species die out eventually? Given the initial population is 200, the model is described by the recursion formula  $p_{n+1} = 0.8p_n + 100$ ,  $p_0 = 200$ . Investigating the time series plot and table on the TI-92 produces the following:

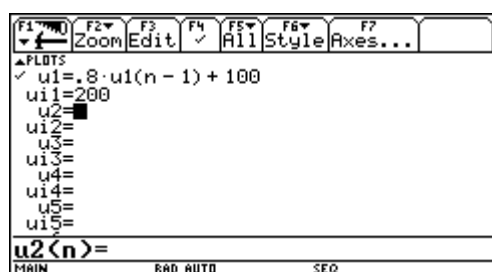


Figure 1. Model:

$$p_{n+1} = 0.8p_n + 100, p_0 = 200$$

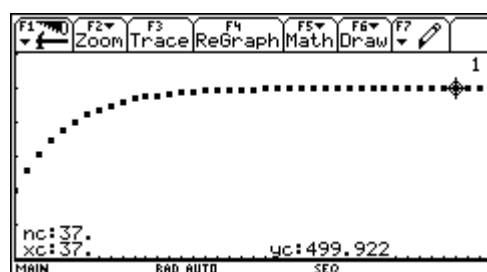


Figure 2. Time series plot of model

$$p_{n+1} = 0.8p_n + 100, p_0 = 200$$

The time series plot (Figure 2) illustrates graphically that the population **appears to be** leveling off or approaching a limit of about 500.

F1	F2	F3	F4	F5	F6	F7
Plot	Setup	Cell	Header	Calc	Util	Stat
n	u1					
0.	200.					
1.	260.					
2.	308.					
3.	346.4					
4.	377.12					
5.	401.7					
6.	421.36					
7.	437.09					
n=0.						
MAIN RAD AUTO SEQ						

Figure 3. Table for n = 0, ..., 7

F1	F2	F3	F4	F5	F6	F7
Plot	Setup	Cell	Header	Calc	Util	Stat
n	u1					
33.	499.81					
34.	499.85					
35.	499.88					
36.	499.9					
37.	499.92					
38.	499.94					
39.	499.95					
40.	499.96					
n=40.						
MAIN RAD AUTO SEQ						

Figure 4. Table for n = 33, ..., 40

From also observing the iterations on the table (Figures 3 and 4), it **appears** that after 30 generations the population is leveling off or approaching a **limit** of about 500.

Add another column to the table. Take the absolute value of the difference between the current population and the **apparent limit**, 500, (a reasonable guess).

F1	F2	F3	F4	F5	F6	F7
Plot	Setup	Cell	Header	Calc	Util	Stat
DATA	Generation	Population	500-Pop			
	c1	c2	c3			
1	0	200.	300.			
2	1	260.	240.			
3	2	308.	192.			
4	3	346.4	153.6			
5	4	377.12	122.88			
6	5	401.696	98.304			
7	6	421.357	78.6432			
c1.Title="Generation"						
MAIN RAD AUTO SEQ						

Figure 5. Table for n = 0, ..., 6

F1	F2	F3	F4	F5	F6	F7
Plot	Setup	Cell	Header	Calc	Util	Stat
DATA	Generation	Population	500-Pop			
	c1	c2	c3			
8	7	437.085	62.9146			
9	8	449.668	50.3316			
10	9	459.735	40.2653			
11	10	467.788	32.2123			
12	11	474.23	25.7698			
13	12	479.384	20.6158			
14	13	483.507	16.4927			
Br14c1=13						
MAIN RAD AUTO SEQ						

Figure 6. Table for n = 7, ..., 13

F1	F2	F3	F4	F5	F6	F7
Plot	Setup	Cell	Header	Calc	Util	Stat
DATA	Generation	Population	500-Pop			
	c1	c2	c3			
29	28	499.42	.580284			
30	29	499.536	.464228			
31	30	499.629	.371382			
32	31	499.703	.297106			
33	32	499.762	.237684			
34	33	499.81	.190148			
35	34	499.848	.152118			
Br35c1=34						
MAIN RAD AUTO SEQ						

Figure 7. Table for n = 28, ..., 34

F1	F2	F3	F4	F5	F6	F7
Plot	Setup	Cell	Header	Calc	Util	Stat
DATA	Generation	Population	500-Pop			
	c1	c2	c3			
35	34	499.848	.152118			
36	35	499.878	.121694			
37	36	499.903	.097356			
38	37	499.922	.077884			
39	38	499.938	.062308			
40	39	499.95	.049846			
41	40	499.96	.039877			
Br41c1=40						
MAIN RAD AUTO SEQ						

Figure 8. Table for n = 34, ..., 40

When will the population be within a tolerance  $\varepsilon = 20$  of 500? That is, when will  $|p_n - 500| < 20$ ? Looking at the table (Figure 6), for  $n \geq 13$ ,  $|p_n - 500| < 20$ . We say a sequence of numbers,  $p_1, p_2, p_3, \dots$  approaches a **limit**  $L$  ( $\lim_{n \rightarrow \infty} p_n = L$ ), if for every tolerance  $\varepsilon > 0$ , there is some  $N$  such that for every  $n \geq N$ ,  $|p_n - L| < \varepsilon$ . From your experimental evidence it appears that the population is approaching a limit of 500. How can you know for sure that this continues? Use the definition of limit of a sequence to verify that 500, (the equilibrium point of the linear discrete dynamical system) is the limit of the sequence  $p_n$  as  $n$  goes to infinity.

The population stabilizes (reaches an **equilibrium point**) whenever the population remains the same after subsequent generations. In other words,

$p_n = p_{n+1} = p_{n+2} = p_{n+3} \dots = x$ . Mathematically speaking, an **equilibrium point** of the recursion formula  $p_n = f(p_{n-1})$  is a solution to the equation  $x = f(x)$ . In this case  $x = .8x + 100$ . Therefore, the equilibrium point is 500. After much experimentation, students can be guided to conjecture and later prove that if a discrete dynamical system has a limit, then the limit is an equilibrium point.

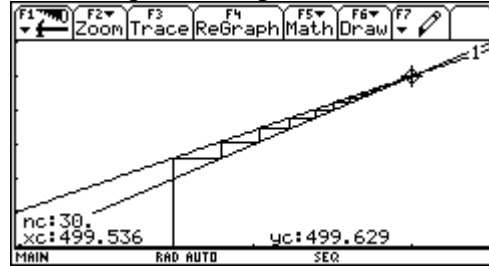


Figure 9. Cobweb plot

The **cobweb plot** (Figure 9) exhibits the relationship between the input (number of inhabitants in the current generation) on the horizontal axis and the output (number of inhabitants in the next generation) on the vertical axis. Observe that in this model the population will be very close to the equilibrium point after many generations. It is clear that  $\lim_{n \rightarrow \infty} p_n = 500$ . The sequence is "staircasing in" toward the equilibrium point. In this case the equilibrium point is "**attracting**".

In exponential models the population multiplier remains constant. However, many factors such as weather or competition with another species can affect the population multiplier. To get a more realistic model we need to replace the constant population multiplier,  $R$ , with a function,  $R(p)$ , that depends on the population. Let  $R(p) = a(1 - bp)$  where  $a$  and  $b$  are constants. This is the **logistic population model**  $p_{n+1} = a(1 - bp_n)p_n$  where positive constants  $a$  and  $b$  describe the underlying biology. The constant  $a$  determines the value of the population multiplier when the population is close to zero and there is very little competition for resources and  $b$  determines how quickly the population multiplier falls as the population rises depending on the amount of resources such as food, water and shelter. Given the model and an initial condition  $p_1 = k$ , a positive constant, students can calculate  $p_2, \dots, p_{20}$  and draw a time series graph. They can be instructed to try to connect the biology represented by different values of  $p_1$  and of the constants  $a$  and  $b$  in the model with the behavior exhibited by the models when describing the results. Figures 10 and 11 illustrate that the model  $p_{n+1} = 2.7(1 - 0.001p_n)p_n$ ,  $p_0 = 50$  predicts that the population seems to be approaching a limit near 630.

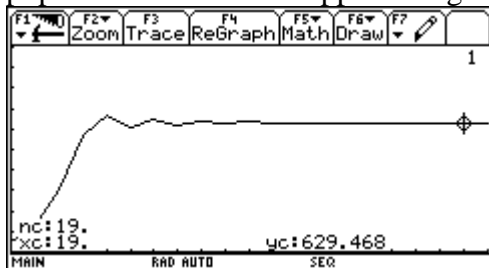
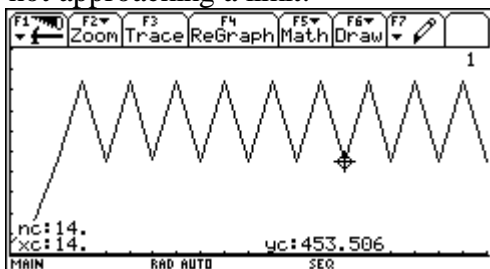


Figure 10. Time Series Plot

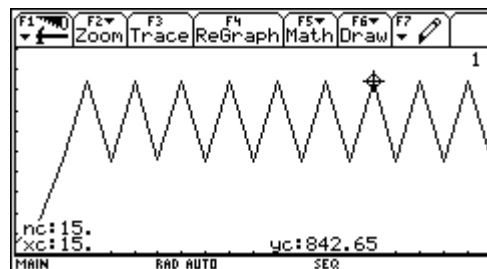
F1	F2	F3	F4	F5	F6	F7
Setup	Cell	Window	Del	Row	Int	Row
n	u1					
13.	628.25					
14.	630.59					
15.	628.96					
16.	630.1					
17.	629.3					
18.	629.86					
19.	629.47					
20.	629.74					
n=13.						
MAIN	RAD	AUTO	SEQ			

Figure 11. Table for  $n = 13, \dots, 20$

Figures 12-15 illustrate that the model  $p_{n+1} = 3.4 (1 - 0.001 p_n) p_n$ ,  $p_0 = 50$  predicts that the population seems to be a 2-cycle bouncing between approximately 453 and 843 and not approaching a limit.



**Figure 12.** Time series plot of model  $p_{n+1} = 3.4 (1 - 0.001 p_n) p_n$ ,  $p_0 = 50$



**Figure 13.** Time series plot of model  $p_{n+1} = 3.4 (1 - 0.001 p_n) p_n$ ,  $p_0 = 50$

F1	F2	F3	F4	F5	F6	F7
Setup	Cell	Head	Del	Pos	Int	Pos
n	u1					
0.	50.					
1.	161.5					
2.	460.42					
3.	844.67					
4.	446.08					
5.	840.11					
6.	456.69					
7.	843.62					
n=7.						

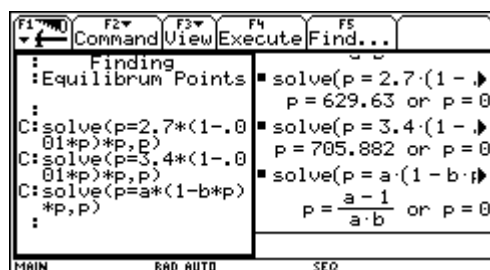
**Figure 14.** Table for  $n = 0, \dots, 7$

F1	F2	F3	F4	F5	F6	F7
Setup	Cell	Head	Del	Pos	Int	Pos
n	u1					
13.	841.49					
14.	453.51					
15.	842.65					
16.	450.81					
17.	841.77					
18.	452.85					
19.	842.44					
20.	451.29					
n=13.						

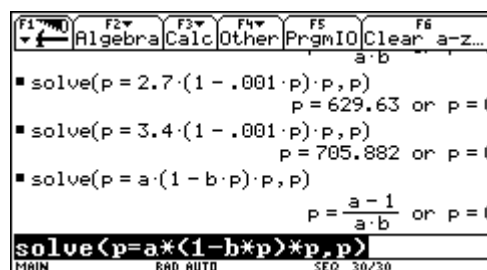
**Figure 15.** Table for  $n = 13, \dots, 20$

Students can work in groups using time series and cobweb analysis to investigate what differences the coefficient  $a$  causes in the logistic population model  $f(p) = ap_n (1 - bp_n)$ . They can be guided to make conjectures and, eventually, prove them. This learning technique has been very effective in my classes.

The TI-92 includes not only the graphing and table features, but also a computer algebra system. These capabilities facilitate investigating equilibrium points. For example, it can solve the equation  $p = a (1 - bp) p$  symbolically as well as draw the cobweb plot graph showing the two intersection points of  $y = a (1 - bp) p$  and  $y = p$ .

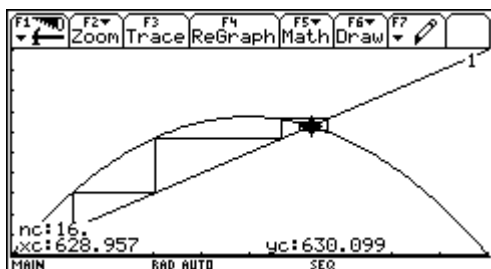


**Figure 16.** Script

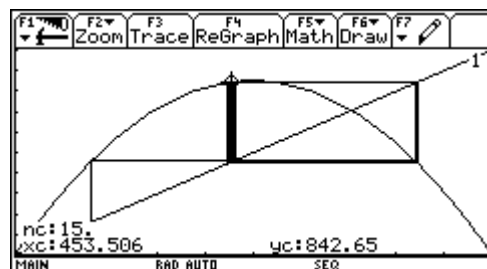


**Figure 17.** Solving symbolically

The analytic solution is displayed in Figures 16 and 17. Figure 16 portrays how lab exercises can be set up as a script file in the text editor with executable commands. Furthermore, the cobweb plot portrays whether the equilibrium point is attracting or repelling.



**Figure 18.** Cobweb plot of model  
 $p_{n+1} = 2.7 ( 1 - 0.001 p_n ) p_n$ ,  $p_0 = 50$



**Figure 19.** Cobweb plot of model  
 $p_{n+1} = 3.4 ( 1 - 0.001 p_n ) p_n$ ,  $p_0 = 50$

In Figure 17 the cobweb pictures the population climbing stairs, spiraling in on the nonzero equilibrium point and approaching the limit near 630. We call this an attracting equilibrium point. On the other hand, Figure 18 reveals a 2-cycle bouncing between approximately 453 and 843 and not approaching a limit. Therefore it is not an attracting equilibrium point. It is repelling. These results agree with previous observations made examining the time series plots. At this point, after experimenting with many models, students make the conjecture that if the logistic population model approaches a limit, then the limit is an equilibrium point.

During the course of the semester students can return to the logistic population model as a discrete dynamical system to motivate the study of the derivative and composition of functions. Later in the year they can also extend the study to continuous dynamical systems.

## REFERENCES

- Connors, Mary Ann. (in press) "Exploring the Logistic Population Model as a Discrete Dynamical System Using the TI-92," *Proceedings of the Tenth International Conference on Technology in Collegiate Mathematics*. Chicago, Illinois November 6-9, 1997.
- Connors, Mary Ann. (1997, Spring). "Technology in Mathematics Education: A Personal Perspective," *Journal of Computing in Higher Education*. 8(2), pp. 94-108.
- Connors, Mary Ann. (1997). "Exploring the Logistic Population Model," *International Journal of Computer Algebra in Mathematics Education*, Volume 4, Number 1.
- Wattenberg, Frank. (1995). *Calculus in a Real and Complex World*. Boston: PWS-Kent.